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# COMBINATORIAL APPROXIMATION TO THE DIVERGENCE OF ONE-FORMS ON SURFACES

BY

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ABSTRACT

We consider the approximation of a differential operator on forms by combinatorial objects via the correspondences of Whitney and de Rham. We prove that the Hilbert space dual of the combinatorial coboundary is an $L^2$ approximation to the codifferential of one-forms on a two-dimensional Riemannian manifold.

**1. Whitney's inner product in cochain spaces**: The de Rham cohomology of a smooth compact manifold $M$ is identified with the simplicial cohomology of a triangulation through integration of forms over images of simplices. This procedure to associate a cochain with each differential form is now known as de Rham's mapping [4, par. 21].

Whitney [9] has constructed a partial inverse to this mapping using barycentric coordinates. If $\mu_p$ denotes the barycentric coordinate with respect to a vertex $p$ in a triangulation ($\mu_p$ is a continuous piecewise linear but nondifferentiable function on the manifold, supported by the star of $p$), then the Whitney mapping has the following effect on a $q$-simplex $[p_0, \dots, p_q]$:

$$W[p_0, \dots, p_q] = q! \sum_{j=0}^{q} (-1)^j \mu_{p_j} d\mu_{p_0} \wedge \dots \wedge d\hat{\mu}_{p_j} \wedge \dots \wedge d\mu_{p_q}$$

where $\hat{}$ over a symbol means deletion. This defines $W$ completely by linear extension. If $R$ denotes de Rham's mapping, then we have $RW = \mathrm{id}$ on the finite-dimensional linear space of $q$-cochains in the triangulation.

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Now suppose $M$ is equipped with a Riemannian metric. Then there is a natural definition of a finite measure space carried by the Borel $\sigma$-algebra of $M$, and a natural Laplace operator on smooth functions. The volume measure on $M$ together with the metric on each fibre of $\Lambda^q T^*M$ make the set of sections of this bundle into an inner product space by

$$(\omega, \psi) := \int_M \langle \omega, \psi \rangle_p dV(p).$$

The completion of this inner product space is denoted $L^2\Lambda^q T^*M$.

The *Hodge Laplacian* of a differential form is the generalization of the Laplacian obtained as the unbounded $L^2$-operator $d^*d + dd^*$, where $d$ is the usual alternating differential (e.g. [4, §4]) and duality is taken in the sense of $L^2\Lambda^q T^*M$.

Dodziuk and Patodi [5][6] used the mappings $R$ and $W$ to obtain combinatorial approximations to the Hodge Laplacian on forms. For a given triangulation $\tau$ of $M$ they consider the combinatorial coboundary $d_\tau$ associated with $\tau$. $d_\tau$ is a linear operator on cochains; its effect on a $q$-simplex $[p_0, \dots, p_q]$ is

$$d_\tau[p_0, \dots, p_q] = \sum\nolimits_p' [p, p_0, \dots, p_q]$$

where $p$ runs over the set of all vertices of $\tau$ such that $[p, p_0, \dots, p_q]$ is a $(q+1)$-simplex of $\tau$.

Whitney's mapping $W$ is injective and thus defines an inner product on the space of $q$-cochains of $\tau$ by

$$(c_1, c_2) := \int_M \langle Wc_1, Wc_2 \rangle_p dV(p).$$

Eckmann remarked [7] that any inner product in cochain spaces gives rise to a combinatorial Hodge theory, provided the coboundary is defined as the Hilbert space dual of the boundary. We prefer to follow the convention of [5] and define the "boundary" $d^*$ as the $l^2$ dual of the combinatorial coboundary. This has no topological consequences in the case of a compact manifold.

We now quote some results from [5].

2. PROPOSITION: *The Whitney mapping commutes with the differential, i.e., $dWc = Wd_\tau c$ for every cochain $c$.*

Here the differential of $Wc$ should be taken on the interior of the $(n-1)$-skeleton, and then interpreted as being almost everywhere defined. Proposition 2 says that $W$ is a map of complexes.

3. THEOREM: *Let $f$ be a $C^\infty$ $q$-form on $M$. There exists a constant $C_f$ such that*

$$\sup_{p \in M} |f(p) - WRf(p)| \leq C_f \sup_{\sigma \in K} \operatorname{diam} \sigma.$$

Here $K$ is the simplicial complex of the triangulation $\tau$. It follows from the above two results that $Wd_\tau Rf$ is a uniform approximation to $df$.

4. **Subdivision of a triangulation:** The results of Dodziuk and Patodi can be applied to obtain arbitrarily close combinatorial approximations of the differential of a form, if one has an iterative procedure of subdividing a simplicial complex $K$ such that the maximum diameters of the simplices become small enough but angles remain bounded away from zero. Whitney [9] has described such a method for complexes of arbitrary dimension. For two-dimensional complexes, Albeverio and Zegarlinski [1] describe a simpler method, which they call the **regular standard subdivision**. It consists of cutting edges in half, and dividing each triangle in four smaller ones with the same shape as the original one.

It was kindly suggested by an anonymous referee that the subdivision scheme of [3] may be helpful for generalizing the result of the present paper to higher dimensions and higher rank forms.

Dodziuk and Patodi also claim that $d_\tau^*$ is *not* a good approximation to $d^*$, giving a counter example for one-forms in $\mathbb{R}^2$ [6, appendix 2]. It turns out that their example is not valid because of an illegal use of duality, neglecting boundary terms when they are not negligible. In fact, the present paper aims to show

5. THEOREM: *Let $\tau : K \to M$ be a triangulation of a compact two-dimensional Riemannian manifold $M$ by a finite simplicial complex $K$. Suppose $\tau$ is regular in the sense that on each 2-simplex $\sigma$ of $K$, $\tau$ is a $C^\infty$ mapping with respect to barycentric coordinates, the derivatives of which extend continuously to the boundary of $|\sigma|$.*

*Let $f \in \Gamma\Lambda^1 T^* M$ be a smooth one-form. Let $S^n\tau : S^n K \to M$ denote the $n$-th regular standard subdivision of $\tau$ with corresponding operators $W_n$, $R_n$ and $d_n$. Then $W_n d_n^* R_n f$ converges to $d^* f$ in $L^2(M, dV)$.*

$d^*$ is called the **divergence operator** on forms.

Dodziuk was considering this approximation to prove convergence of the eigenvalues of the combinatorial Laplacian $d_n^* d_n + d_n d_n^*$ to those of the Hodge Laplacian. He and Patodi proved this in [6]. All these were partial results towards the solution of the Ray-Singer conjecture on the equality of analytic and

Reidemeister-Franz torsion. The latter problem and its solution are treated in [2] and [8].

We shall first prove a result similar to Theorem 5 for flat $\mathbf{R}^2$. The same techniques apply to flat tori and cylinders. The proof of Theorem 5 is given as a separate section but relies heavily on the flat case.

We shall prove Theorem 5 in four steps. The first step will treat the special case of Dodziuk's and Patodi's counter example. As a preliminary we isolate a technical lemma on matrix invertibility.

6. DEFINITION: *Let $V$ be an index set and let $M$ be a $V \times V$ matrix. By $\mathcal{D}(M)$ we denote the diagonal part of $M$, and $\mathcal{N}(M) = M - \mathcal{D}(M)$.*

We say that $M$ is ***dominated by its diagonal*** if it has nonnegative entries and if for all $v \in V$:

$$M(v,v) \geq \sum_{w \neq v} M(v,w). \tag{1}$$

Condition (1) is equivalent to the requirement that the sum of the entries in the $v$-th row in $\mathcal{D}(M)^{-1}\mathcal{N}(M)$ be less than or equal to 1.

7. LEMMA: *Let $V$ be an index set and let $M$ be a $V \times V$ matrix which is dominated by its diagonal. If for some odd $n$, the diagonal of the matrix $Q^n = (\mathcal{D}(M)^{-1}\mathcal{N}(M))^n$ is bounded away from zero, then $M$ is invertible, and the inverse is given by*

$$M^{-1} = (1 - Q + Q^2 - \ldots + Q^{n-1})(1 + Q^n)^{-1}\mathcal{D}(M)^{-1}. \tag{2}$$

*Proof:* If $M$ is dominated by its diagonal, then so is $\mathcal{D}(M)^{-1}M$. Moreover,

$$\begin{cases} \mathcal{D}(\mathcal{D}(M)^{-1}M) &= 1, \\ \mathcal{N}(\mathcal{D}(M)^{-1}M) &= \mathcal{D}(M)^{-1}\mathcal{N}(M) = Q. \end{cases}$$

Therefore, it is no restriction to suppose $\mathcal{D}(M) = 1$, which we do implicitly from here until the end of the proof.

All the matrices we are considering have nonnegative entries only; hence their norm as $l^\infty$-operators can be measured by application to the constant function 1. Since $M$ is dominated by its diagonal, $Q$ is a contraction, and so are $Q^n$ and $\mathcal{N}(Q^n)$.

In fact, the norm of $\mathcal{N}(Q^n)$ is *strictly* less than 1 since $\mathcal{D}(Q^n)$ is bounded away from zero by hypothesis. Suppose $\mathcal{D}(Q^n) \geq \varepsilon > 0$ and put $P = (Q^n - \varepsilon)/(1+\varepsilon)$. Then $1 + Q^n$ is invertible, its inverse being given by the geometric series

$$(1 + Q^n)^{-1} = \frac{1}{1+\varepsilon}(1 - P + P^2 - \cdots).$$

For odd $n$ we have

$$1 + Q^n = (1 + Q)(1 - Q + Q^2 - \cdots + Q^{n-1}).$$

Therefore $M$ itself is invertible, and its inverse is given by (2). □

Next we give a graph theoretical interpretation of this lemma. We consider an unoriented graph as an ordered pair $G = (V, E)$ where $V$ is a collection of vertices, and $E$ a collection of edges, i.e., pairs in $V$. The ***adjacency matrix*** $A$ of $G$ is the symmetric $V \times V$-matrix having entry 1 in position $(v, w)$ if $v \neq w$ and $\{v, w\} \in E$, 0 in all other cases.

**8.** COROLLARY: *Let $k \in \mathbb{N}$ and let $G = (V, E)$ be an unoriented graph where each vertex has at most $k$ neighbours. If there are $n, R \in \mathbb{N}$ such that each vertex in $V$ is less than $R$ steps away from a cycle of odd length at most $n$, and if we write $A$ for the adjacency matrix of $G$, then $k + A$ is invertible and the inverse is a bounded operator in $l^p(V)$, $1 \leq p \leq \infty$.*

*Proof:* The $j$-th power of $A$ has the following interpretation: $(A^j)_{v,w}$ is the number of paths of length $j$ from $v$ to $w$, where the convention is that a path may run along the same edge several times. Suppose $n$ is odd (otherwise replace it by $n + 1$).

The $(2R+n)$-th power of $A$ has entries larger than or equal to 1 everywhere on the diagonal, by the hypothesis on cycles. Now apply Lemma 7 with $M = k + A$. $l^p$-boundedness follows from (2) and from the fact that $\mathcal{D}(M)^{-1} = 1/k$. □

In order to treat piecewise affine triangulations of the plane, we need formulae for the scalar product of cochains in dimensions 0 and 1.

**9.** LEMMA: *Let $\tau : K \to \mathbb{R}^2$ be a regular triangulation of $\mathbb{R}^2$ that is affine in every 2-simplex of $K$. Consider $\mathbb{R}^2$ as a Riemannian manifold with its standard metric. Then the following rules hold for scalar products of cochains in dimension 0:*

(a) *the square of the norm of any single vertex is one sixth of the surface area of its star,*
(b) *the scalar product of adjacent edges is one twelfth of the surface area of their stars' intersection,*
(c) *all other pairs of vertices are orthogonal.*

*Proof:* The simplices of $K$ can be considered subsets of $\mathbb{R}^2$. Then all three claims follow directly from the fact that $W(p_{r,s}) = \mu_{p_{r,s}}$, the barycentric coordinate of $p_{r,s}$. □

10. DEFINITION: *Consider three independent points $a_0$, $a_1$ and $a_2$ in $\mathbb{R}^2$ with coordinates $(x_0, y_0)$, $(x_1, y_1)$ and $(x_2, y_2)$, respectively. The* **triangulation generated by** $(a_0, a_1, a_2)$ *is constructed as follows. The vertices are $\{p_{r,s}; r, s \in \mathbf{Z}\}$, where $p_{r,s} = a_0 + r(a_1 - a_0) + s(a_2 - a_0)$.*

The 1-simplices are

$$A_{r,s} = [p_{r,s}, p_{r+1,s}]; \quad B_{r,s} = [p_{r,s}, p_{r,s+1}]; \quad C_{r,s} = [p_{r,s+1}, p_{r+1,s}].$$

The 2-simplices are the triangles thus obtained, with an arbitrary but fixed orientation.

The triangulation mapping is the canonical embedding of each simplex.

The surface area of the triangles is equal to half the absolute value of

$$\alpha(a_0, a_1, a_2) = \begin{vmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{vmatrix}.$$

11. LEMMA: *The scalar products of 1-simplices in the triangulation generated by the points $a_0(x_0, y_0)$, $a_1(x_1, y_1)$ and $a_2(x_2, y_2)$ follow from the formulae below after permutation of the indices.*

$$|A_{r,s}|^2 = \frac{1}{6|\alpha|}\big((x_0 - x_2)^2 + (x_0 - x_2)(x_1 - x_2) + (x_1 - x_2)^2 + (y_0 - y_2)^2 + (y_0 - y_2)(y_1 - y_2) + (y_1 - y_2)^2\big),$$

$$(A_{r,s}, C_{r,s}) = \frac{1}{12|\alpha|}\big((x_0 - x_2)^2 - (x_1 - x_2)(x_1 - x_0) + (y_0 - y_2)^2 - (y_1 - y_2)(y_1 - y_0)\big) \tag{3}$$

*Proof:* This follows directly from the definition of the Whitney mapping. □

We are now ready to treat the flat case of Theorem 5.

12. PROPOSITION: *Consider for each natural number $n$ the triangulation of $\mathbb{R}^2$ generated by $(0,0)$, $(0, \frac{1}{n})$ and $(\frac{1}{n}, 0)$ as in Definition 10. We denote by $W_n$, $R_n$ and $d_n^*$ respectively the Whitney and de Rham mappings and the dual of the coboundary in our triangulation. Let $f \in \Lambda^1\mathbb{R}^2$ be infinitely differentiable with compact support. Then $W_n d_n^* R_n f$ converges uniformly to $\delta f$ as $n \to \infty$.*

*Proof:* The form $f$ can be written as $\varphi(x, y)dx + \psi(x, y)dy$. By $xy$-symmetry of our particular triangulation, it is sufficient to prove the claim in the special case $\psi = 0$.

We maintain the notations $p_{r,s}$, $A_{r,s}$, $B_{r,s}$ and $C_{r,s}$ from Definition 10. Then

$$(R_n f)(A_{r,s}) = \int_{x=r/n}^{(r+1)/n} \varphi(x, \frac{s}{n})dx,$$
$$(R_n f)(B_{r,s}) = 0,$$
$$(R_n f)(C_{r,s}) = \int_{t=0}^{1/n} \varphi(\frac{r}{n} + t, \frac{s+1}{n} - t)dt.$$

For the triangulation we are considering, $\alpha = 1/n^2$. From Lemma 9 we get

(a) the squared norm of a vertex is $1/2n^2$,

(b) the scalar product of adjacent vertices is $1/12n^2$,

(c) other pairs of vertices are orthogonal.

For the 1-simplices Lemma 11 gives

(a) the squared norm of 1-simplices $A_{r,s}$ and $B_{r,s}$ is $2/3$,

(b) the squared norm of 1-simplices $C_{r,s}$ is $1/3$,

(c) the scalar product of $A_{r,s}$ with $B_{r,s}$ is $1/6$,

(d) the scalar product of $A_{r,s}$ with $B_{r+1,s-1}$ is $1/6$,

(e) all other pairs of 1-simplices are orthogonal.

The description of the effect of $d_n^*$ on $R_n f$ is somewhat more complicated. Since it is the dual of $d_n$, we have for each 1-simplex $S$ and each vertex $p_{r,s}$

$$\begin{aligned}(d_n^* S, p_{r,s}) &= (S, d_n p_{r,s}) \\ &= (S, A_{r-1,s} - A_{r,s} + B_{r,s-1} - B_{r,s} + C_{r-1,s} - C_{r,s-1}).\end{aligned} \tag{4}$$

The 0-cochain $d_n^* S$ can be written as a (possibly infinite) linear combination of vertices $p_{i,j}$. We interpret (4) as an infinite system of linear equations, indexed by $(r, s) \in \mathbf{Z}^2$, in the unknown coefficients of the linear combination. In matrix form this reads

$$M\mathbf{d} = \mathbf{k}. \tag{5}$$

Remember that the system has $\mathbf{Z}^2$ unknowns, so the columns $\mathbf{d}$ and $\mathbf{k}$ have $\mathbf{Z}^2$ elements, and $M$ is a $\mathbf{Z}^2 \times \mathbf{Z}^2$-matrix !

We observe some properties of the linear system (5). First of all, the column vector $\mathbf{k}$ on the right hand side has nonzero elements only in a finite number of positions $(r, s)$. Next, the infinite matrix $M$ by which the unknowns are multiplied, is symmetric and has only finitely many nonzero elements in each row/column; in fact, its element in position $((i,j),(r,s))$ is

$$M_{(i,j)(r,s)} = (p_{i,j}, p_{r,s}) = \frac{1}{2n^2}\mathrm{Id}_{(i,j)(r,s)} + \frac{1}{12n^2}A_{(i,j)(r,s)}$$

where $A$ is the adjacency matrix of the infinite planar graph corresponding to the 1-skeleton of the triangulation. $A_{(i,j)(r,s)} = 1$ if $p_{r,s}$ and $p_{i,j}$ are neighbours, zero otherwise.

This planar graph has the property that each vertex is part of a cycle of length three and has exactly six neighbours. It follows from Corollary 8 that $M$ is invertible, and the inverse, when considered as an operator in $l^\infty(\mathbf{Z}^2)$, is given by a norm convergent geometric series. On rescanning the proof of Lemma 7, it appears that the series can take the following form.

$$M^{-1} = 2n^2\left(1 - \frac{A}{6} + \frac{A^2}{36}\right) \times \frac{18}{19}(1 - P + P^2 - \dots)$$

where $P$ is convolution by a function on $\mathbb{Z}^2$ that:

is nowhere negative;

is finitely supported;

has integral 17/19.

The matrix $P$ is obtained by subtracting from $A^3$ its diagonal element 1/18, and then dividing the result by 19/18.

The next step in our calculation consists of writing down the column vector $\mathbf{k}$, since that is what we have to multiply by $M^{-1}$ to get the unknown $\mathbf{d}$. Since $R_n f$ is zero on all of the $B_{u,v}$, we only need to consider the cases $S = C_{u,v}$ and $S = A_{u,v}$.

If $S = C_{u,v}$, then the right hand side of (4) is

$$\begin{cases} 1/3 & \text{if } (u,v) = (r-1,s); \\ -1/3 & \text{if } (u,v) = (r,s-1); \\ 0 & \text{otherwise.} \end{cases}$$

If $S = A_{u,v}$, then the right hand side of (4) is

$$\begin{cases} 5/6 & \text{if } (u,v) = (r-1,s); \\ -5/6 & \text{if } (u,v) = (r,s); \\ 1/6 & \text{if } (u,v) = (r,s-1); \\ -1/6 & \text{if } (u,v) = (r-1,s+1); \\ 0 & \text{otherwise.} \end{cases}$$

From this it follows that we can write $d_n^* R_n f$ as

$$2n^2(1 - A/6 + A^2/36)(1 + A^3/216)^{-1}\mathbf{k} \tag{6}$$

where

$$\mathbf{k}_{r,s} = \{\frac{1}{3}(R_n f)(C_{r-1,s}) - \frac{1}{3}(R_n f)(C_{r,s-1}) + \frac{5}{6}(R_n f)(A_{r-1,s}) \\ - \frac{5}{6}(R_n f)(A_{r,s}) + \frac{1}{6}(R_n f)(A_{r,s-1}) - \frac{1}{6}(R_n f)(A_{r-1,s+1})\}. \tag{7}$$

Finally, $W_n d_n^* R_n f$ is the piecewise linear extension of the above expression (6) to the whole of $\mathbb{R}^2$.

Now let $\varepsilon > 0$ be given, we shall find an $n_0 \in \mathbb{N}$ such that, for $n \geq n_0$, $\|W_n d_n^* R_n f - d^* f\|_\infty < \varepsilon$. First choose $k$ such that

$$\sum_{k'=k}^{\infty} \left(\frac{19}{18}\|P\|\right)^{k'} < \frac{\varepsilon}{8\|\nabla\varphi\|_\infty}$$

(the matrix between brackets is $\mathcal{N}(A^3)$). The expression $\mathbf{k}_{r,s}$ in (7) is

$$\begin{aligned}
&\frac{1}{3}\int_{t=0}^{1/n}\left[\varphi(\frac{r-1}{n}+t,\frac{s+1}{n}-t)-\varphi(\frac{r}{n}+t,\frac{s}{n}-t)\right]dt \\
&\quad+\frac{5}{6}\int_{t=0}^{1/n}\left[\varphi(\frac{r-1}{n}+t,\frac{s}{n})-\varphi(\frac{r}{n}+t,\frac{s}{n})\right]dt \\
&\quad+\frac{1}{6}\int_{t=0}^{1/n}\left[\varphi(\frac{r}{n}+t,\frac{s-1}{n})-\varphi(\frac{r-1}{n}+t,\frac{s+1}{n})\right]dt \\
&=\frac{1}{3n}\int_{t=0}^{1/n}\left[\left(-\frac{\partial\varphi}{\partial x}+\frac{\partial\varphi}{\partial y}\right)(\frac{r}{n}+t,\frac{s}{n}-t)\right]dt+O(\frac{1}{n^3}) \\
&\quad+\frac{5}{6n}\int_{t=0}^{1/n}\left[\left(-\frac{\partial\varphi}{\partial x}\right)(\frac{r}{n}+t,\frac{s}{n})\right]dt+O(\frac{1}{n^3}) \\
&\quad+\frac{1}{6n}\int_{t=0}^{1/n}\left[\left(\frac{\partial\varphi}{\partial x}-2\frac{\partial\varphi}{\partial y}\right)(\frac{r}{n}+t,\frac{s}{n})\right]dt+O(\frac{1}{n^3}) \\
&=-\frac{1}{n^2}\frac{\partial\varphi}{\partial x}(\frac{r}{n},\frac{s}{n})+O(\frac{1}{n^3}).
\end{aligned}$$

The effect of $A/6$ and $19P/17$ is to average a function over close neighbours of a vertex in the graph. Thus, for example, application of $A/6$ to $\mathbf{k}$, evaluated at $(r,s)$, yields $-1/n^2$ times a weighted average of $\partial\varphi/\partial x$ over seven points in the rectangle $((r\pm1)/n,(s\pm1)/n)$. If $k' \leq k$, then $19P/17$ is an average over neighbours not further away than $3k$ edges. It follows from the infinite differentiability of $\varphi$ that

$$\|(\frac{19}{17}P)^{k'}\frac{\partial\varphi}{\partial x}-\frac{\partial\varphi}{\partial x}\|_\infty = O(k/n).$$

Similarly,

$$2n^2(1-A/6+A^2/36)\times\frac{18}{19}\sum_{k'=0}^{k}(-P)^{k'}\{\cdots\}=-2\times\frac{1}{2}\frac{\partial\varphi}{\partial x}\left(1-(\frac{17}{19})^k\right)+O(k^2/n).$$

But $d^*(\varphi(x,y)dx) = -\frac{\partial\varphi}{\partial x}$. The original claim then follows by making $k$ large and choosing $n$ so large that $O(k^2/n) < \varepsilon/3$. □

We now generalize the above result to triangulations of $\mathbb{R}^2$ which are affine transformations of the one treated before.

13. PROPOSITION: *Let $a_0(x_0, y_0)$, $a_1(x_1, y_1)$ and $a_2(x_{2,2})$ be independent points in $\mathbb{R}^2$ and consider for every natural number $n$ the triangulation of $\mathbb{R}^2$ generated by $a_0$, $a_0 + (a_1 - a_0)/n$ and $a_0 + (a_2 - a_0)/n$. We denote by $W_n$, $R_n$ and $d_n^*$ respectively the Whitney and de Rham mappings and the dual of the coboundary in this triangulation. Let $f \in \Lambda^1\mathbb{R}^2$ be infinitely differentiable with compact support. Then $W_n d_n^* R_n f$ converges uniformly to $d^* f$ as $n \mapsto \infty$.*

*Proof:* Again, we assume for simplicity $f = \varphi(x,y)dx$ and we use the notations $p_{r,s}$, $A_{r,s}$ etc. Let $(x_{r,s}, y_{r,s})$ be the coordinates of $p_{r,s}$. This time the effect of de Rham's mapping is given by

$$(R_n f)(A_{r,s}) = (x_1 - x_0)\int_{t=0}^{1}\varphi(x_{r,s} + t(x_1 - x_0), y_{r,s} + t(y_1 - y_0))dt,$$

$$(R_n f)(B_{r,s}) = (x_2 - x_0)\int_{t=0}^{1}\varphi(x_{r,s} + t(x_2 - x_0), y_{r,s} + t(y_2 - y_0))dt,$$

$$(R_n f)(C_{r,s}) = (x_2 - x_1)\int_{t=0}^{1}\varphi(x_{r,s} + x_2 - x_0 + t(x_1 - x_2), y_{r,s} + y_2 - y_0 + t(y_1 - y_2))dt.$$

Formula (4) is still valid, and the inversion problem for $M$ is nearly the same as before, after replacing the factor $1/2n^2$ by $1/2\alpha n^2$ wherever appropriate. This follows from Lemma 9.

The column $\mathbf{k}$ which has to be multiplied on the left by $M^{-1}$ is slightly more complicated.

$$
(8) \quad
\begin{aligned}
(Rf, dp_{r,s}) = {} & Rf(A_{r-1,s} - A_{r,s})(|A_{r,s}|^2 + (A_{r,s}, B_{r,s}) + (A_{r,s}, C_{r,s})) \\
& + Rf(B_{r,s-1} - B_{r,s})(|B_{r,s}|^2 + (A_{r,s}, B_{r,s}) - (B_{r,s}, C_{r,s})) \\
& + Rf(C_{r-1,s} - C_{r,s-1})(|C_{r,s}|^2 + (A_{r,s}, C_{r,s}) - (B_{r,s}, C_{r,s})) \\
& + Rf(A_{r,s-1} - A_{r-1,s+1})((A_{r,s}, B_{r,s}) - (A_{r,s}, C_{r,s})) \\
& + Rf(B_{r-1,s} - B_{r+1,s-1})((A_{r,s}, B_{r,s}) + (B_{r,s}, C_{r,s})) \\
& + Rf(C_{r-1,s-1} - C_{r,s})((A_{r,s}, C_{r,s}) + (B_{r,s}, C_{r,s})).
\end{aligned}
$$

Denote by $O(2^{-kn})$ any term converging to zero at least as quickly as $2^{-kn}$ as $n \to \infty$. The symbol $\sim$ will be used hereafter for equality op to $O(2^{-3n})$. From the infinite differentiability of $\varphi$ it follows that

(9)

$$
\begin{aligned}
Rf(A_{r-1,s} - A_{r,s}) &\sim (x_1 - x_0)\left(-(x_1 - x_0)\frac{\partial\varphi}{\partial x} - (y_1 - y_0)\frac{\partial\varphi}{\partial y}\right), \\
Rf(B_{r,s-1} - B_{r,s}) &\sim (x_2 - x_0)\left(-(x_2 - x_0)\frac{\partial\varphi}{\partial x} - (y_2 - y_0)\frac{\partial\varphi}{\partial y}\right), \\
Rf(C_{r-1,s} - C_{r,s-1}) &\sim (x_1 - x_2)\left(-(x_1 - x_2)\frac{\partial\varphi}{\partial x} - (y_1 - y_2)\frac{\partial\varphi}{\partial y}\right), \\
Rf(A_{r,s-1} - A_{r-1,s+1}) &\sim (x_1 - x_0)\left((x_0 + x_1 - 2x_2)\frac{\partial\varphi}{\partial x} + (y_0 + y_1 - 2y_2)\frac{\partial\varphi}{\partial y}\right), \\
Rf(B_{r-1,s} - B_{r+1,s-1}) &\sim (x_2 - x_0)\left((x_0 + x_2 - 2x_1)\frac{\partial\varphi}{\partial x} + (y_0 + y_2 - 2y_1)\frac{\partial\varphi}{\partial y}\right), \\
Rf(C_{r-1,s-1} - C_{r,s}) &\sim (x_1 - x_2)\left((2x_0 - x_1 - x_2)\frac{\partial\varphi}{\partial x} + (2y_0 - y_1 - y_2)\frac{\partial\varphi}{\partial y}\right),
\end{aligned}
$$

where the partial derivatives are understood to be evaluated at $p_{r,s}$. A lengthy but straightforward computation using Lemma 11 then yields

$$
(d_n^* R_n f, p_{r,s}) = -\alpha\frac{\partial\varphi}{\partial x}(p_{r,s}) + O(2^{-3n}).
$$

Application of $M^{-1}$ is the same as in Proposition 12, with $n^2$ replaced by $n^2\alpha$. The convergence follows. □

The third step is to treat more general triangulations of flat $\mathbb{R}^2$. We shall still suppose that the triangulation is locally finite and that the individual triangles have surface area bounded below.

14. PROPOSITION: *Let $\tau$ be a locally finite, piecewise linear triangulation of the plane. Suppose the 2-simplices of $\tau$ are mapped to triangles having surface area bounded from below. Denoting by $W_n$, $R_n$ and $d_n^*$ respectively the Whitney and de Rham mappings and the dual of the combinatorial coboundary in the $n$-th regular standard subdivision of $\tau$, then for every smooth one-form $f$ with compact support the following hold:*

(i) *On compact subsets of the interior of triangles for $\tau$, the functions $W_n d_n^* R_n f$ converge uniformly to $d^* f$ as $n \to \infty$;*
(ii) *$W_n d_n^* R_n f$ is uniformly bounded in $n$.*
*In particular,*

$$W_n d_n^* R_n f \to d^* f \quad in \ L^2(\mathbb{R}^2).$$

*Proof:* (i) After a few iterations of the subdivision, the interiors of the original triangles are tiled like any part of the plane in Proposition 13, except that the value of $\alpha$ is not everywhere the same (but bounded below by hypothesis). It follows that on these interiors

$$(d_n^* R_n f, p) = -\alpha \partial\varphi/\partial x(p) + O(\alpha 2^{-2n}). \tag{10}$$

Furthermore the matrix $M$ of vertex scalar products is dominated by its diagonal: the inverse can be computed by a power series as in Lemma 9.

We want to verify convergence on (compact subsets $K$ of) the interior of the original triangles. The inverse $M^{-1}$ corrects for $\alpha$ in (10) as follows. Up to a term $O(1)$, $M^{-1}$ is a norm convergent sum of matrices giving local weighted averages (with sum of the weights $\alpha^{-1}$). If we cut the infinite sum after the $k$-th term where the tail is sufficiently small, and then perform the subdivision sufficiently often ($n$ times), then on $K$ the combinatorial $d_n^*$ turns out to be a uniform (strong) approximation of the differential operator $d^*$.

(ii) Now consider the 'boundary' points, i.e., points of the 1-skeleton of the original triangulation $\tau$.

First suppose $p$ is a *vertex* of $\tau$. Number the 1-simplices belonging to the star of $p$ as $A_1, \ldots, A_n$ in counterclockwise order, and let $A_j$ be oriented such that $p$ appears with a plus sign in the boundary of $A_j$. Call $B_j$ the 1-simplex connecting the other endpoints of $A_j$ and $A_{j+1}$, where the convention is to apply modulo $n$ arithmetic on indices $j$. For a fixed orthonormal coordinate system of the plane centered at $p$, let $(x_j, y_j)$ be the coordinates of the other endpoint of $A_j$. Let $\alpha_j$ be twice the oriented surface area of the triangle formed by $A_j$, $A_{j+1}$ and $B_j$.

Then it follows from the definition of the Whitney mapping and the scalar product of cochains that

$$\begin{aligned}(d^*Rf,p) &= (Rf,\sum_{j=1}^{n} A_j)\\ &= \sum_{j=1}^{n} Rf(A_j)|A_j|^2 + (Rf)(A_{j-1})(A_j,A_{j-1}) + Rf(A_{j+1})(A_j,A_{j+1})\\ &\quad + Rf(B_{j-1})(A_j,B_{j-1}) + Rf(B_j)(A_j,B_j)\\ &= \sum_{j=1}^{n} \frac{1}{6\alpha_j}\{Rf(A_j)\,[(x_j-x_{j+1})(x_j-2x_{j+1}) + (y_j-y_{j+1})(y_j-2y_{j+1})]\\ &\quad + Rf(A_{j+1})\,[(x_j-x_{j+1})(2x_j-x_{j+1}) + (y_j-y_{j+1})(2y_j-y_{j+1})]\\ &\quad + Rf(B_j)\,[(x_j-x_{j+1})(x_j+x_{j+1}) + (y_j-y_{j+1})(y_j+y_{j+1})]\}\end{aligned}$$

Note that under $n$-fold regular standard subdivision, all coordinates are divided by the same power of two.

If, in the above expression, we replace $f$ by its linear approximation (first order Taylor series) near $p$, then the error will be $O(2^{-3n})$ for the $n$-th regular standard subdivision of the triangulation. Furthermore, by linearity it is no restriction to suppose that

$$f = (\varphi_0 + \varphi_1 x + \varphi_2 y)dx + \text{second order terms}.$$

Substituting the linear approximation gives

$$\begin{aligned}(d^*Rf,p) \sim \sum_{j=1}^{n} \frac{1}{6\alpha_j} &\{[\varphi_0 x_j + \varphi_1 x_j^2/2 + \varphi_2 x_j y_j]\\ &\quad\times [(x_j-x_{j+1})(x_j-2x_{j+1}) + (y_j-y_{j+1})(y_j-2y_{j+1})]\\ &+ [\varphi_0 x_{j+1} + \varphi_1 x_{j+1}^2/2 + \varphi_2 x_{j+1} y_{j+1}]\\ &\quad\times [(x_j-x_{j+1})(2x_j-x_{j+1}) + (y_j-y_{j+1})(2y_j-y_{j+1})]\\ &+ [\varphi_0(x_{j+1}-x_j) + \varphi_1(x_{j+1}^2-x_j^2)/2 + \varphi_2(x_{j+1}y_{j+1}-x_jy_j)]\\ &\quad\times [(x_j-x_{j+1})(x_j+x_{j+1}) + (y_j-y_{j+1})(y_j+y_{j+1})]\}.\end{aligned}$$

We treat the homogeneous parts separately. Terms of the form $\varphi_0 x^3$ vanish for each $j$ separately. Terms $\varphi_0 xy^2$ yield

$$\sum_{j=1}^{n} \frac{y_j-y_{j+1}}{6\alpha_j}\,[-3y_{j+1}x_j + 3y_jx_{j+1}] = -\frac{1}{2}\sum_{j=1}^{n}(y_j-y_{j+1}) = 0.$$

It follows that $d_n^* R_n f$, and therefore $W_n d_n^* R_n f$, is uniformly essentially bounded in a neighbourhood of $p$. Since the points which lie ultimately in the 0-skeleton of an iterated regular subdivision of $\tau$ are dense in $\mathbb{R}^2$, $W_n d_n^* R_n f$ is uniformly bounded on compact subsets of $\mathbb{R}^2$. Remember that $f$ has compact support, and that the matrix $M$ in section 12 had a bounded inverse in $l^\infty$: therefore $W_n d_n^* R_n f$ is uniformly bounded in all of $\mathbb{R}^2$, even near the skeleton of the triangulation $\tau$. □

**15.** *Proof of Theorem 5:* It suffices to prove that the sequence converges uniformly on compact subsets of the interior of the original, unrefined triangulation, and is uniformly bounded near every point of $M$.

Let $p$ be an interior point. Choose local orthonormal coordinates originating at $p$ that are an affine transformation of the barycentric coordinates everywhere else in the 2-simplex to which $p$ belongs. Without restriction suppose that in these coordinates $f(x,y) = \varphi(x,y)dx$.

Regular subdivisions of $\tau$ near $p$ have the familiar periodic graph structure from Propositions 12 and 13. Thus (8) holds. The approximation (9) is equally valid, provided the symbol $\sim$ is interpreted as equality up to terms of order $O(2^{-3n})$ as before.

The scalar products of vertices and edges, however, cannot be computed exactly anymore. They are approximately equal to the values from Lemmas 9 and 11, where the order of approximation is $O(2^{-n})$ for $|A|^2$ and $(A,C)$, and $O(2^{-3n})$ for $|p|^2$ and $(p,q)$.

It follows that near $p$, i.e., for vertices $p_{r,s}$ at most a fixed number of 1-simplices away from $p$ in the 1-skeleton of the triangulation

$$(d_n^* R_n f, p_{r,s}) = -\alpha(p)\frac{\partial\varphi}{\partial x}(p_{r,s}) + O(2^{-3n}).$$

The finite square matrix of scalar products $(p,q)$ of vertices has entries of order $O(2^{-2n})$ that are given, up to order $O(2^{-3n})$, by the formulae

$$|q|^2 \sim \frac{1}{6} \times \text{ total surface area of } \mathrm{St}(q), \text{ the star of } q,$$
$$(p,q) \sim \frac{1}{12} \times \text{surface area of } \mathrm{St}(p) \cap \mathrm{St}(q).$$

For sufficiently large $n$ the inversion procedure of lemma 7 can be applied to obtain that $d_n^* R_n f(p_{r,s})$ is close to an average of $-\partial\varphi/\partial x$ over a small neighbourhood of $p_{r,s}$. Since $W_n$ is the piecewise linear extension operator (that is, functions in its range are linear in barycentric coordinates on the interior of every triangle), we have again that $W_n d_n^* R_n f \to f$ uniformly in a neighbourhood of $p$.

Similarly, if $p$ is an arbitrary point of $M$, an argument as in Proposition 14 produces a neighbourhood of $p$ on which $W_n d_n^* R_n f$ is uniformly bounded in $n$.

□

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## References

1. S. Albeverio and B. Zegarlinski, *Construction of convergent simplicial approximations of quantum fields on Riemannian manifolds*, University of Bochum preprint SFB 237, 1989.
2. J. Cheeger, *Analytic torsion and Reidemeister torsion*, Proc. Natl. Acad. Sci. USA **74** (1977), 2651–2654.
3. J. Cheeger, W. Müller and R. Schrader, *On the curvature of piecewise flat spaces*, Comm. Math. Phys. **92** (1984), 405–454.
4. G. de Rham, *Variétés différentiables*, Hermann, Paris, 1960.
5. J. Dodziuk, *Finite-difference approach to the Hodge theory of harmonic forms*, Amer. J. Math. **98** (1976), 79–104.
6. J. Dodziuk and V.K. Patodi, *Riemannian structures and triangulations of manifolds*, J. Indian Math. Soc. **40** (1976), 1–52.
7. B. Eckmann, *Harmonische Funktionen und Randwertaufgaben in einem Komplex*, Comment. Math. Helv. **17** (1945), 240–255.
8. W. Müller, *Analytic torsion and R-torsion of Riemannian manifolds*, Adv. Math. **28** (1978), 233–305.
9. H. Whitney, *Geometric Integration Theory*, Princeton University Press, Princeton, 1957.